

On conformally invariant CLE explorations

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Abstract

We study some conformally invariant dynamic ways to construct the Conformal Loop Ensembles with simple loops introduced in earlier papers by Sheffield, and by Sheffield and Werner. One seemingly interesting outcome is a conformally invariant way to measure a distance of a CLE loop to the boundary “within” the CLE, when one identifies all points of each loop.

1 Introduction

The present paper will be devoted to the study of some properties of the Conformal Loop Ensembles (CLE) defined and studied by Scott Sheffield in [9] and by Sheffield and Werner in [11].

A simple CLE can be viewed as a random countable collection $(\gamma_j, j \in J)$ of disjoint simple loops in the unit disk that are non-nested (almost surely no loop surrounds another loop in CLE). The paper [11] shows that there are several different ways to characterize them and to construct them. In that paper, a CLE is defined to be such a random family that also possesses two important properties: It is conformally invariant (more precisely, for any fixed conformal map Φ from the unit disk onto itself, the law of $(\Phi(\gamma_j), j \in J)$ is identical to that of $(\gamma_j, j \in J)$ – this allows to define the law of the CLE in any simply connected domain via conformal invariance) and it satisfies a certain natural restriction property that one would expect from interfaces in physical models. It is shown in [11] that there exists exactly a one-parameter family of such CLEs. Each CLE law corresponds exactly to some $\kappa \in (8/3, 4]$ in such a way that for this κ , the loops in the CLE are loop-variants of the SLE_κ processes (these are the Schramm-Loewner Evolutions with parameter κ – recall that an SLE_κ for $\kappa \leq 4$ is a simple curve with Hausdorff dimension $1 + \kappa/8$), and that conversely, for each κ in that range, there exists exactly one corresponding CLE. Part of the arguments in the paper [11] are based on the analysis of discrete “exploration algorithms” of these loop ensembles, where one slices the CLE open from the boundary and their limits (roughly speaking when the step-size of explorations tends to zero).

In the earlier paper [9], Sheffield had constructed explicitly a number of random collections of loops, using variants of SLE_κ processes. In particular, for any $\kappa \in (8/3, 4]$, he has shown how to construct random collections of SLE_κ -type loops that should be the only possible candidates for the conformally invariant scaling limit of various discrete models, or of level lines of certain continuous models. The rough idea is to choose some boundary point x on the unit circle (“the root”) and from there to launch some branching exploration

tree of SLE processes (or rather target-independent variants of the SLE process called the $SLE(\kappa, \kappa - 6)$ processes) that will trace some loops along the way, that one keeps track of. For each κ and x , there are in fact several ways to do this. One particular way, that we will refer to as symmetric in the present paper, is to impose certain “left-right” symmetry in the law of the exploration tree, but several other possibilities are described in [9]. Hence, for each κ , the exploration tree is defined via the choice of the root x and the exploration “strategy” that describes how “left-right” asymmetric the exploration is. These exploration strategies are particularly natural, because they are invariant under all conformal transformations that preserve x . Note also that it is conjectured that the exploration indeed traces a tree with continuous branches, but that this fact is not proved (to our knowledge). However, it is known that they indeed trace continuous loops along the way. So to sum up, once κ , x and a given strategy are chosen, the Loewner differential equation enables to construct a random family of loops in the unit disc (and the law of this family a priori depends on κ , on x and on the chosen strategy). One can also note that the symmetric strategy is very natural for $\kappa = 4$ from the perspective of the Gaussian Free Field, but much less so from the perspective of interfaces of lattice models. For instance, in the case of $\kappa = 3$ viewed as the scaling limit of the Ising model (see [2]), the “totally asymmetric” procedure seems more natural.

Recall that when one works directly in the SLE-framework, certain questions turn out to be rather natural – it is for instance possible to derive rather directly the values of certain critical exponents, to compute explicitly probabilities of certain events, or to study questions related to conformal restriction – while the setting of the Loewner equation does not seem so naturally suited for some other questions. Proving reversibility of the SLE path (that the random curve defined by an SLE from a to b is the same as that defined by an SLE from b to a) turns out to be very tricky, see [13, 5].

Based on the conjectured or proved relation to discrete models and to the Gaussian Free Field, Sheffield conjectures in [9] that for any given $\kappa \leq 4$, all the random collections of loops traced by the various exploration trees have the same law.

One consequence of the results of Sheffield and Werner in [11] is that this conjecture is indeed true for all symmetric explorations: For each $\kappa \in (8/3, 4]$, the law of the random collection of loops traced by a symmetric $SLE(\kappa, \kappa - 6)$ exploration tree rooted at x does not depend on x . In fact, their common law is that of “the CLE” with SLE_κ -type loops and it can also be constructed (see [11]) using clusters of Brownian loop-soups. Note that a by-product of the proof is another derivation of the reversibility of the corresponding SLE. One main idea in [11] is to study the asymptotic behavior of the discrete explorations when the steps get smaller and smaller, and to prove that it converges to the above-mentioned symmetric $SLE(\kappa, \kappa - 6)$ process.

The first main result of the present paper can be summarized as follows: *For each $\kappa \in (8/3, 4]$, all the random collections of SLE_κ -type loops constructed via Sheffield’s asymmetric exploration trees in [9] have the same law. They all are the CLE_κ ’s constructed in [11].* The proof of this fact will heavily rely on the results of [11], but we will try to make our paper as self-contained as possible.

The above-mentioned constructions of CLE are interesting because they are conformally invariant. In order to define them, one needs to choose a starting point and an asymmetry parameter, but one does not have to choose a target point.

The second main point of our paper is to highlight something specific to the case $\kappa = 4$ i.e. to CLE_4 (recall that this is the CLE that is most directly related to the Gaussian Free Field, see [7, 6, 3, 10]). In this particular case, it is possible to define a conformally invariant and unrooted (one does not need to even choose a starting point) growing mechanism of loops (the term “exploration” that is used in this paper is a little bit misleading, as it is not proved that the growth process is in fact a deterministic function of the CLE, we will discuss this at the end of the paper). Roughly speaking, the growth process that progressively discovers loops is growing “uniformly” from the boundary (even if it is a Poisson point process and each loop is discovered at once) and does not require to choose a root. The fact that such a conformally invariant non-local growth mechanism exists at all is quite surprising (and the fact that its time-parametrization as seen from different points does exactly coincide even more so). It also leads to a conformal invariant way to describe distances between loops in a CLE (where any two loops in a CLE are at a positive distance of each other) that will be studied in more detail in the subsequent paper [12], and to open questions that we will describe at the end of the paper.

2 Background and first main statement

In the present section, we recall some ideas, arguments and results from [9, 11], and set up the framework that will enable us to derive our main results in a rather simple way.

2.1 Bessel processes and principal values

Suppose throughout this section that $\delta \in (0, 1]$. It is easy to define the *squared Bessel process* $(Z_t, t \geq 0)$ of dimension δ started from $Z_0 = z_0 \geq 0$ as the unique solution to the stochastic differential equation (SDE in short)

$$dZ_t = 2\sqrt{Z_t}dB_t + \delta dt$$

where $(B_t, t \geq 0)$ is a Brownian motion (note that it is implicit that this solution is non-negative because one takes its square root).

The non-negative process $Y_t = \sqrt{Z_t}$ is then usually called the Bessel process of dimension δ started from $\sqrt{z_0}$. It is not formally the solution to the SDE

$$dY_t = dB_t + (\delta - 1)\frac{dt}{2Y_t}$$

because it gets an (infinitesimal) upwards push whenever it hits the origin, so that $Y_t - (\delta - 1)\int_0^t ds/(2Y_s)$ is not a martingale (note for instance that when $\delta = 1$, the process Y_t is a reflected Brownian motion, which is clearly not the solution to $dY_t = dB_t$). This SDE however describes well the evolution of Y while it is away from the origin, and if one adds the fact that Y is almost surely non-negative, continuous and that the Lebesgue measure of $\{t > 0 : Y_t = 0\}$ is almost surely equal to 0, then it does characterize Y uniquely. Note that the filtration generated by Y and by B do coincide (B can be recovered from Y).

Bessel processes have the same scaling property as Brownian motion: When $Y_0 = 0$, then for any given positive ρ , $(Y_t, t \geq 0)$ and $(\rho^{-1}Y_{\rho^2 t}, t \geq 0)$ have the same law (this is an immediate consequence of the definition of its squared process Z). Just as in the case of the Itô measure on Brownian excursions, it is possible to define an infinite measure λ on (positive) Bessel excursions of dimension δ . An excursion e is a continuous function $(e(t), t \in [0, \tau])$ defined on an interval of non-prescribed length $\tau = \tau(e)$ such that $e(0) = e(\tau) = 0$ and $e(t) > 0$ when $t \in (0, \tau)$.

Standard excursion theory shows that it is possible to define the process Y by glueing together a Poisson point process $(e_u, u \geq 0)$ of these Bessel excursions of dimension δ . Furthermore, the fact that $t \mapsto (Y_t)^{2-\delta}$ is a local martingale when Y is away from the origin (which follows immediately from the definition of Y and from Itô's formula) shows readily that

$$\lambda(\{e : \sup_{s \leq \tau} e(s) \geq x\}) = cx^{\delta-2}$$

for some constant c that can be chosen to be equal to one (this is the normalization choice of λ).

Suppose that we are given a parameter $\beta \in [-1, 1]$ and that for each excursion e of the Bessel process Y , one tosses an independent coin in order to choose $\epsilon(e) = \epsilon_\beta(e) \in \{-1, 1\}$ in such a way that the probability that $\epsilon(e) = +1$ (respectively $\epsilon(e) = -1$) is $(1+\beta)/2$ (resp. $(1-\beta)/2$). Then, one can define a process $X^{(\beta)}$ by glueing together the excursions $(\epsilon(e) \times e)$ instead of the excursions (e) . Note that $|X^{(\beta)}| = X^{(1)} = Y$, but some of the excursions of $X^{(\beta)}$ are negative as soon as $\beta < 1$. The process $X^{(0)}$ is the *symmetrized* Bessel process such that $X^{(0)}$ and $-X^{(0)}$ have the same law. Note that (as opposed to $Y = X^{(1)}$) the process $X^{(\beta)}$ is not a deterministic function of the underlying driving Brownian motion B in the SDE when $\beta \in (-1, 1)$, because additional randomness is needed to choose the signs of the excursions. However, B is still a martingale with respect to the filtration generated by $X^{(\beta)}$ because the signs of the excursions are in a way independent of the excursions.

In the sequel, it will be useful to consider the processes $X^{(\beta)}$ for various values of β simultaneously. Clearly, it is easy to first define Y and then to couple all signs in such a way that for each excursion e of Y , $\epsilon_\beta(e) \geq \epsilon_{\beta'}(e)$ as soon as $\beta \geq \beta'$; we will implicitly always work with such a coupling.

In the context of SLE processes, it turns out to be essential to try to make sense of a quantity of the type $\int_0^t ds/X_s^{(\beta)}$. Simple considerations make it possible to check that $i(e) := \int_0^\tau ds/e(s)$ is finite for λ -almost all excursions, and that $\lambda(i(e)1_{i(e) \geq 1})$ is finite as well. Note also that the scaling shows that

$$\lambda(\{e : i(e) \geq x\}) = x^{\delta-2} \lambda(\{e : i(e) \geq 1\}).$$

It follows that typically, the number of excursions that occur before time 1 for which $i(e) \in [2^{-n}, 2^{-n+1})$ is of the order of $(2^{-n})^{\delta-2}$, so that their cumulative contribution to $\int_0^t ds/Y_s$ is of the order of $(2^{-n})^{\delta-1}$. If we sum this over n , one readily sees that when $t > 0$, then

$$\int_0^t ds/Y_s = \infty$$

almost surely as soon as $\delta \leq 1$ (and this argument can be easily made rigorous) due to the cumulative contributions of the many short excursions during the interval $[0, t]$. Hence, $\int_0^t ds/X_s^{(\beta)}$ can not be defined as a simple absolutely converging integral.

There are however ways to circumvent this difficulty. The first classical one works for all $\delta \in (0, 1]$ but it is specific to the case where $\beta = 0$ i.e. to the symmetrized Bessel process $X^{(0)}$. In that case, when one formally evaluates the cumulative contribution to $\int_0^t ds/X_s^{(0)}$ of the excursions for which $i(e) \in [2^{-n}, 2^{-n+1})$, then the central limit theorem suggests that one will get a value of the order of $2^{-n} \times (2^{(2-\delta)n})^{1/2} = 2^{-\delta n/2}$; when one then sums over n , one gets an almost surely converging series. This heuristic can be easily be made rigorous, and this shows that one can define a process $I_t^{(0)}$ that one can informally interpret as $\int_0^t ds/X_s^{(0)}$ (even though this last integral does not converge absolutely). Another possible way to characterize this process is that it is the only process such that:

- $t \mapsto I_t^{(0)}$ is almost surely continuous and satisfies Brownian scaling.
- $dI_t^{(0)} - dt/X_t^{(0)}$ is zero on any time-interval where $X^{(0)}$ is non-zero.
- The process $I^{(0)}$ is a deterministic function of the process $X^{(0)}$.

Let us reformulate and detail our first approach to $I_t^{(0)}$ in a way that will be useful for our purposes. Suppose that $r > 0$ is given and small. We denote by J_r the set of times that belong to an excursion of Y away from the origin, that has time-length at least r^2 (we choose r^2 in order to have the same scaling properties as for the height and $i(e)$). Then, because the integral $\int ds/e(s)$ on each individual excursion is finite, we see that it is possible to define without any difficulty the absolutely converging integral

$$I_t^{(0,r)} := \int_0^t ds 1_{s \in J_r} / X_s^{(0)}.$$

Then, as $r \rightarrow 0$ the continuous process $I^{(0,r)}$ converges to the continuous process $I^{(0)}$. More rigorously

Lemma 1. *When $n \rightarrow \infty$, then on any compact time-interval, the sequence of continuous functions $I^{(0,1/2^n)}$ converges almost surely to a limiting continuous function $I^{(0)}$.*

Proof. Let $\tau = \tau(r_0)$ denote the end-time of first excursion that has time-length at least r_0^2 (here r_0 should be thought off as very large, so that this time, which is greater than r_0^2 , is large too). It suffices to prove the almost sure convergence on the interval $[0, \tau]$ (as any given compact interval is inside some interval $[0, \tau]$ for small r_0).

Let us suppose that $n \geq m$. Notice that the process $t \mapsto I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)}$ is monotonous (i.e. non-increasing or non-decreasing) on each excursion of Y , so that

$$\sup_{t \leq \tau} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2 = \sup_{t \leq \tau, Y_t=0} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2.$$

Next we define the σ -field \mathcal{F}_0 generated by the knowledge of all excursions $|e|$, but not their signs. If we condition on \mathcal{F}_0 and look at the value of $I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)}$ at the end-times

of the excursions of length greater than 2^{-n} , we get a discrete martingale. From Doob's L^2 inequality, we therefore see that almost surely,

$$E \left(\sup_{t \leq \tau} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2 \mid \mathcal{F}_0 \right) \leq 4E \left((I_\tau^{(0,1/2^n)} - I_\tau^{(0,1/2^m)})^2 \mid \mathcal{F}_0 \right).$$

The right-hand side is in fact the mean of the square of a series of symmetric random variables of the type $\sum \epsilon_j i_j$ for some given i_j and coin-tosses ϵ_j . Therefore, it is equal to $4 \sum_e i(e)^2$ where the sum is over all excursions appearing before time τ , corresponding to times in $J_{1/2^n} \setminus J_{1/2^m}$. By simple scaling, the expectation of this quantity is equal to a constant times $2^{-m\delta} - 2^{-n\delta}$, so that finally

$$E \left(\sup_{t \leq \tau} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2 \right) \leq C(2^{-m\delta} - 2^{-n\delta}) \leq C2^{-m\delta}.$$

It then follows easily (via Borel-Cantelli) that almost surely, the function $t \mapsto I_t^{(0,1/2^n)}$ converges uniformly as $n \rightarrow \infty$ on the time-interval $[0, \tau]$ (and that the limiting process $I^{(0)}$ is continuous). \square

This construction of $I^{(0)}$ can not be directly extended to the case where $\beta \neq 0$. Indeed, the cumulative contributions of those excursions of $X^{(\beta)}$ for which $i(e) \in [2^{-n}, 2^{1-n})$ is then of the same order of magnitude than when $\beta = 1$ (the previously described case where one looks at the integral of ds/Y_s). A solution when the dimension of the Bessel process is smaller than 1, is to compensate the explosion of this integral appropriately. Let us first describe this in the case where $\beta = 1$ (i.e. $X^{(\beta)} = Y$ is the non-negative Bessel process). As for instance explained in [9], Section 3, it is possible to characterize the principal value $I_t = I_t^{(1)}$ of the integral of $1/Y_t$ as the unique process such that:

- $t \mapsto I_t$ is almost surely continuous.
- $dI_t - dt/Y_t$ is zero on any time-interval where Y is non-zero.
- (I_t, Y_t) is adapted to the filtration of Y and satisfies Brownian scaling.

Let us describe how to construct explicitly this process I_t . For any very small r , recall the definition of the time-set J_r , and define $N_r(t)$ as the number of excursions of time-length at least r^2 that Y has completed before time t . Simple scaling considerations show that (for fixed t), $N_r(t)$ will explode like (some random number times) $r^{\delta-2}$ as $r \rightarrow 0$.

Just as before, there is no problem to define the absolutely converging integral

$$\int_0^t \frac{1_{s \in J_r} ds}{Y_s}.$$

But, as we have already indicated, when $\beta \neq 0$ this quantity tends to ∞ when r tends to 0. One option is therefore to consider the quantity

$$K_t^r := \int_0^t \frac{1_{s \in J_r} ds}{Y_s} - CrN_r(t)$$

where

$$Cr := \frac{\lambda(i(e)1_{\tau(e) \geq r^2})}{\lambda(1_{\tau(e) \geq r^2})}$$

is the mean value of the integral of $1/e$ for an excursion conditioned to have length greater than r^2 . Note that

$$C = \lambda(i(e)1_{\tau(e) \geq 1})/\lambda(1_{\tau(e) \geq 1})$$

is a constant that does not depend on r . When $r \rightarrow 0$, $rN_r(t)$ explodes like $r^{\delta-1}$, but nevertheless:

Lemma 2. *As $n \rightarrow \infty$, the process $K^{1/2^n} = (K_t^{1/2^n}, t \geq 0)$ does almost surely converge uniformly on any compact time-interval to some continuous limiting process $I^{(1)}$.*

Proof. The proof goes along similar lines as in the case $\beta = 0$, but there are some differences. Let us prove again almost sure convergence on each $[0, \tau]$ for each given r_0 . For each $r > r'$, if we condition on \mathcal{F}_0 and follow $t \mapsto K_t^r - K_t^{r'}$ only at end-times of the excursions of time greater than r' , we get a discrete martingale, and therefore, via Doob's inequality and scaling, just as before, we get that

$$E \left(\sup_{t \leq \tau: Y_t = 0} (K_t^r - K_t^{r'})^2 \right) \leq cr_0^{2-\delta} r^\delta.$$

It follows that almost surely, for any t_0 , the function $K^{1/2^n}$ converges uniformly as $n \rightarrow \infty$ on the set $\{t \leq t_0 : Y_t = 0\}$. It follows immediately from the definition of K^r that this uniform convergence takes place on all of $[0, t]$ (just because one adds always the same function on each of the excursions).

On the other hand, if we slightly modify K^r on each excursion interval by adding a linear function that makes it continuous on the closed support of the excursion in order to compensate the $-Cr$ jump of K^r and the end-time of the excursion, one obtains a continuous function \tilde{K}^r such that $|K_t^r - \tilde{K}_t^r| \leq Cr$ for each t . It follows that almost surely $\tilde{K}^{1/2^n}$ converges uniformly on any compact time-interval to the same limit as $K^{1/2^n}$. As the functions \tilde{K}^r are continuous, it follows that this limit is almost surely a continuous function of time. \square

For any β (and as long as $\delta \in (0, 1)$), the very same idea can be used to define a process $I^{(\beta)}$ associated to $X^{(\beta)}$ instead of Y , as the limit when $r \rightarrow 0$ of the process

$$\int_0^t \frac{1_{s \in J_r} ds}{X_s^{(\beta)}} - Cr\beta N_r(t).$$

Let us describe in more detail a variant of the previous construction that will be useful for our purposes. Suppose that one is working with the coupling of all processes $X^{(\beta)}$ (for fixed $\delta \in (0, 1)$). We then define the process

$$X_t^{(\beta, r)} := X_t^{(\beta)} 1_{t \in J_r} + X_t^{(0)} 1_{t \notin J_r}.$$

In other words, we replace $X^{(\beta)}$ by $X^{(0)}$ on all excursions of length smaller than r^2 . Clearly, this makes it possible to make sense of the continuous process

$$\int_0^t \left(\frac{1}{X_s^{(\beta, r)}} - \frac{1}{X_s^{(0)}} \right) ds$$

(because only the times in J_r i.e. in the macroscopic excursions will contribute). We can therefore define the process

$$I_t^{(\beta,r)} := I_t^{(0)} + \int_0^t 1_{s \in J_r} \left(\frac{1}{X_s^{(\beta,r)}} - \frac{1}{X_s^{(0)}} \right) ds - \beta Cr N_r(t).$$

This process $I^{(\beta,r)}$ follows exactly the evolution of $I^{(0)}$ except that some excursions of length greater than r^2 are sign-changed (and on these excursions $I_t^{(\beta,r)} + I_t^{(0)}$ is constant), and that at the end of each of those excursions, it makes a small jump of $-\beta Cr$.

Then, almost surely, when $r \rightarrow 0$, the process $I^{(\beta,r)}$ converges uniformly on any compact time-interval to a process $I^{(\beta)}$ because the two processes

$$I_t^{(0)} - \int_0^t 1_{s \in J_r} ds / X_s^{(0)}$$

and

$$\int_0^t 1_{s \in J_r} ds / X_s^{(\beta)} - \beta Cr N_r(t) - I_t^{(\beta)}$$

do almost surely uniformly converge to zero on any given compact interval.

The previous definition of $I^{(\beta)}$ can not be directly adapted to the case $\delta = 1$. However, one notes that for any real μ , the process $I_t^{<\mu>} := I_t^{(0)} + \mu \ell_t$, where ℓ is the local time at 0 of X does also satisfy the Brownian scaling property and that $dI_t^{<\mu>} = dt/X_t$ on all intervals where X is non-zero. This process $I^{<\mu>}$ can in fact again be approximated via $N_r(t)$ (using the classical approximation of Brownian local time); more precisely, it is the limit as $r \rightarrow 0$ of the process

$$I_t^{<\mu,r>} := I_t^{(0)} + \mu r N_r(t).$$

2.2 From Bessel processes to branching SLEs

We now briefly recall from [9] how to use the previous considerations in order to define the branching SLE exploration trees. If we work in the upper half-plane \mathbb{H} , then Loewner's construction shows that as soon as one has defined a continuous real-valued function $(w_t, t \geq 0)$, one can define a two-dimensional ‘‘Loewner chain’’ (that in many cases turns out to correspond to a two-dimensional path) as follows: For any $z \in \overline{\mathbb{H}}$, define the solution $(Z_t = Z_t(z))$ to the ordinary differential equation

$$Z_t = z + \int_0^t \frac{2ds}{Z_s - w_s}.$$

This equation is well-defined up to a (possibly infinite) explosion/swallowing time $T(z) = \sup\{t \geq 0 : \inf\{|Z_s - w_s| : s \in [0, t]\} > 0\}$. For each given t , the map $g_t : z \mapsto Z_t(z)$ is a conformal map from some subset of \mathbb{H} onto \mathbb{H} .

When w_t is chosen to be equal to $\sqrt{\kappa} B_t$, where B is a standard real-valued Brownian motion, then this defines the SLE_κ processes, that turn out to be simple curves as soon as $\kappa \leq 4$. The so-called $\text{SLE}(\kappa, \kappa - 6)$ processes are variants of SLE_κ with a particular target

independence property first pointed out in [8]. More precisely, suppose that one considers the joint evolution of two points (W_t, O_t) in \mathbb{R} started from (W_0, O_0) with $W_0 \neq O_0$, and described by

$$dO_t = \frac{2dt}{O_t - W_t} \text{ and } dW_t = \sqrt{\kappa}dB_t + \frac{\kappa - 6}{W_t - O_t}dt \quad (1)$$

as long as $W_t \neq O_t$ (where B is a standard Brownian motion). Then, one can use the random function W as the driving function of our Loewner chain, which is this $\text{SLE}(\kappa, \kappa - 6)$. There is no difficulty in defining the process (W, O) as long as W_t does not hit O_t , but more is needed to understand what happens after such a meeting time.

Note that if one writes $X_t = (W_t - O_t)/\sqrt{\kappa}$, then

$$dX_t = dB_t + \frac{\kappa - 4}{\kappa X_t}dt$$

so that X evolves like a Bessel process of dimension

$$\delta = 3 - \frac{8}{\kappa} \in (0, 1]$$

when $\kappa \in (8/3, 4]$, and that dO_t is a constant multiple of dt/X_t as long as $X_t \neq 0$. Furthermore, the knowledge of X_t and of O_t enables to recover $W_t = O_t + \sqrt{\kappa}X_t$.

This gives the following options to define a driving process $(W_t, t \geq 0)$ at all times (even for $W_0 = O_0 = 0$):

- When $\delta \in (0, 1)$ (i.e., $\kappa \in (8/3, 4)$) and $\beta \in [-1, 1]$: Define first one of the Bessel processes $X^{(\beta)}$ as before started at 0, and its corresponding process $I^{(\beta)}$. Then define $O_t^{(\beta)} = 2\sqrt{\kappa}I_t^{(\beta)}$ and

$$W_t^{(\beta)} = \sqrt{\kappa}X_t^{(\beta)} + O_t^{(\beta)} = \sqrt{\kappa}X_t^{(\beta)} + 2\sqrt{\kappa}I_t^{(\beta)}.$$

- When $\delta = 1$ (i.e., $\kappa = 4$) and $\mu \in \mathbb{R}$, then, define $O_t^{<\mu>} = 4I_t^{<\mu>}$ and

$$W_t^{<\mu>} = 2B_t + O_t^{<\mu>} = 2B_t + 4I_t^{<\mu>}$$

where B is standard one-dimensional Brownian motion.

In all these cases, one constructs a couple (W_t, O_t) that satisfies the Brownian scaling property and that evolves according to (1) when $W_t \neq O_t$. The process W therefore defines a Loewner chain from the origin to infinity in the upper half-plane. The Brownian scaling property shows that this Loewner chain is invariant (in law) under scaling (modulo time-parametrization). This makes it possible to also define (via conformal invariance) the law of the Loewner chain in \mathbb{H} from 0 to some $u \in \mathbb{R}$ (and more generally from any boundary point to any other boundary point of a simply connected domain) by considering the conformal image of the previously defined chain from 0 to infinity under a conformal map from \mathbb{H} onto itself that maps 0 onto itself, and ∞ onto u .

Let us now discuss target-independence. The target-independence property of all these processes is that, up to the first time at which the Loewner chain disconnects u from infinity, the two Loewner chains (from 0 to ∞ , and from 0 to u) have the same law (modulo time-change).

When $O_t - W_t$ is not equal to 0, the fact that the local evolution of the couple (W_t, O_t) is independent of the target point is derived (via Itô formula computations) in [8]. Note that the two evolutions match up to a time-change only (as time corresponds to the size of the Loewner chain seen from either infinity or from u). In order to check that target-independence remains valid at all times, one needs to check that the “local push” rule that is used in order to define $I_t^{(\beta)}$ (or $I_t^{<\mu>}$ when $\kappa = 4$) and then O_t is also the same (modulo the time-change) for both processes. This is basically explained in Section 7 of [9].

This makes it then possible to define a branching “SLE” starting from 0 and aiming at all points on $\overline{\mathbb{R}}$. Suppose for instance that $u \in \mathbb{R}$ is fixed, and start the previous SLE($\kappa, \kappa - 6$) at zero that targets ∞ . Let τ be the first time at which the evolution disconnects u from ∞ (a simple 0 – 1-law argument can be used to show that this time is in fact almost surely finite). At this time, it is easy to check (for instance by contradiction) that $O_\tau = W_\tau$. Until τ , the previous argument shows that it is possible to alternatively view the SLE as targeting u (and stopped at the first time it disconnects ∞ from u). Then, at this time, we can continue by launching two SLE’s independently, one targeting ∞ and one targeting u . In this way, we have coupled the two entire SLE’s, from 0 to ∞ and from 0 to u , in such a way that they coincide until τ . The generalization to more (and therefore countably many too) boundary points is similar.

In fact, the fact that we are launching independent SLE’s after the disconnection time τ (at which $W_\tau = O_\tau$) makes it also possible to target other boundary points, that are in \mathbb{H} but on the boundary of the Loewner hull K_τ . In this way, we see that it is in fact possible to define an branching SLE($\kappa, \kappa - 6$) structure that targets all points in $\overline{\mathbb{H}}$ with rational coordinates say (another approach to this targeting of inside points can be obtained via the radial Loewner equation formalism, see for instance [9]).

Once this branching SLE is defined, it is possible to define (in a deterministic manner) the collection of loops in \mathbb{H} . Basically, we define “the” loops as the images under the Loewner transformation of the stretches corresponding to the excursions of X (and its branched off versions); the loop will be traced clockwise or anti-clockwise, depending on the sign of the excursion. For instance, for $\beta = 1$, all loops are traced anti-clockwise. With this definition (see [9]), for each given x , κ and β (or μ), it turns out that the law of the family of loops is invariant under any conformal transformation from \mathbb{H} onto itself that maps x onto itself (which basically follows from the combination of target-independence and scale-invariance). This family of loops has the property that each given point in \mathbb{H} is almost surely surrounded by a loop.

To sum up things, this procedure defines:

- When $\kappa \in (8/3, 4)$, for each $\beta \in [-1, 1]$ and for each boundary point x (corresponding to our choice of starting point in the previous setting) a random family of loops that we can denote by $\text{CLE}_\kappa^\beta(x)$.
- When $\kappa = 4$, for each $\mu \in \mathbb{R}$ and each boundary point x , a random family of loops

that we denote by $\text{CLE}_{4,\mu}(x)$.

In [11], it is proved that the law of a $\text{CLE}_\kappa^0(x)$ does not depend on x , and that the law of $\text{CLE}_{4,0}(x)$ does not depend on x . Furthermore, it is shown that these loop ensembles traced by a symmetric SLE procedure (based on symmetrized Bessel processes) are the only family of loops that satisfy some axiomatic properties.

3 The asymmetric explorations

The present section is devoted to the proof of the following fact (and to its analogue for $\kappa = 4$, Proposition 4):

Proposition 3. *For all given $\kappa \in (8/3, 4)$, the law of $\text{CLE}_\kappa^\beta(x)$ does depend neither on x nor on β .*

Let us first state some further consequences of CLE results from [11].

- We have just explained how to build a symmetric $\text{SLE}(\kappa, \kappa - 6)$ process out of a Bessel process, when $\kappa \in (8/3, 4]$.

If we formally replace the Bessel process X by just one Bessel excursion e , the procedure defines an SLE loop. In other words, let us start with the excursion $(e(s), s \leq \tau)$ and use the driving function

$$w_t := \sqrt{\kappa}e_t + 2\sqrt{\kappa} \int_0^t ds/e(s)$$

to generate the Loewner chain. It is shown in [11] that for almost all (positive) Bessel excursion e (according to the Bessel excursion measure that we have denoted by λ), this defines a simple loop $\gamma(e)$ in the upper half-plane, that starts and ends at the origin. The time-length $\tau(e)$ of the loop corresponds to the half-plane capacity seen from infinity of γ . The infinite measure on loops γ that one obtains when starting from the infinite measure λ on Bessel excursions is referred to as the one-point pinned measure in [11].

A by-product of the characterization/uniqueness of CLE derived in [11] is the fact that the image measure on loops $\gamma(e)$ (i.e. the image measure of λ under $e \mapsto \gamma(e)$) is the same as the image measure on loops defined by $e \mapsto \gamma(-e)$ if one forgets about the time-parametrization of the loops (this is related to the reversibility of SLE paths derived by Zhan [13], one can also view this property as a consequence of reversibility). We will call μ^0 this measure on loops in the upper half-plane (just like in [11]). Note that the time-length $\tau(e)$ is both the half-plane capacity of $\gamma(e)$ and of $\gamma(-e)$ for a given e , but that the two loops $\gamma(e)$ and $\gamma(-e)$ are not the same path (even if their traces have the same law).

Suppose now that we consider a symmetric $\text{SLE}(\kappa, \kappa - 6)$ with driving function W , that we stop at the first time T_1 after which it completes an excursion of length greater than r^2 . This defines a certain family of loops in the unit disc via the procedure described

above. If we are given some sign $\epsilon_1 \in \{-1, +1\}$ (independent of the process W), we can decide to modify the previous process W into another process \hat{W} , by just (maybe) changing the sign of the final excursion before T_1 into ϵ_1 (it may be that we do not have to change it in order to have it equal to ϵ_1). Then, clearly, the law of $(\hat{W}, t \leq T_1)$ is absolutely continuous with respect to that of $(W_t, t \leq T_1)$, and it defines also a family of loops in the unit disk. The previous argument shows in fact that the loops defined by these two processes have exactly the same law (because changing the sign of one final excursion does not change the distribution of the corresponding loop).

- Suppose that T is some stopping time for the driving function $(W_t, t \geq 0)$ of the symmetric SLE($\kappa, \kappa - 6$) process started from the origin in the half-plane. Suppose furthermore that almost surely $W_T = O_T$. Then, we know that the process $(W_{T+t} - W_T, t \geq 0)$ is distributed exactly as W itself, and furthermore, it is independent of $(W_t, t \leq T)$. This means that the conditional law given $(W_t, t \leq T)$ of the non-yet explored loops is just a CLE in the yet-to-be explored domain. This makes it possible to change the starting point of the upcoming evolution, because we know that the law of the loops defined by the branching symmetric SLE($\kappa, \kappa - 6$) is independent of the chosen starting point. In particular, if we consider an increasing sequence of stopping times T_n (and $T_0 = 0$) such that $T_n \rightarrow \infty$ almost surely and $W_{T_n} = O_{T_n}$ for each n , and define the process

$$\hat{W}_t = W_t + cN_t \quad (2)$$

where c is some constant and $N_t = \max\{n \geq 0 : T_n \leq t\}$, the planar loops associated with the excursions intervals of the Bessel process X will be distributed according to loops in a CLE $_\kappa$.

Let us now combine the previous two observations in the case where $\kappa < 4$ (i.e., $\delta < 1$). Suppose now that $r > 0$ is fixed. Consider a symmetrized Bessel process $X^{(0)}$ and the corresponding driving function $W^{(0)}$. Define the stopping times $T_n(r)$ as the end-time of the n -th excursion of length at least r^2 of $|X^{(0)}|$. Up to time $T_1(r)$, we perform the exploration using the driving function $W_t^{(0)}$ (that is defined using the symmetrized Bessel process) except that the sign of the last excursion may have been changed depending on the sign $\epsilon_1(r)$. Note that we have just recalled that this procedure defines exactly loops of a CLE. Note that $T_1(r)$ is the end-time of an excursion and corresponds exactly to the completion of a CLE loop. Then, we force a jump of $-\beta C r 2\sqrt{\kappa}$ of the driving function, and continue from there until time $T_2(r)$ by following the dynamics of $W_t^{(0)}$ (possibly changing the sign of the last excursion before $T_2(r)$). At $T_2(r)$, we again wake a jump of $-\beta C r 2\sqrt{\kappa}$. Combining the previous two items shows that the law of the constructed loops (including the small ones) are exactly those of a CLE (and that if one completes the picture in the non-explored domain by independent CLE's, one would obtain a full CLE).

Let us now look at the driving function of the previously defined Loewner chain. It is exactly the one that would obtain if the signs of the excursions of length larger than r^2 are those given by $\epsilon_1(r), \epsilon_2(r)$ etc., and one also puts in the negative jumps at each times $T_n(r)$. This corresponds exactly to the driving function

$$W_t^{(\beta, r)} := \sqrt{\kappa} X_t^{(\beta, r)} + 2\sqrt{\kappa} I_t^{(\beta, r)}.$$

In order to conclude, we need to control what happens when $r \rightarrow 0$. The law of the loops that are in fact constructed may change, but for each given r , they are always distributed like loops in a CLE, and if one fills in the unexplored pieces with independent CLE, one gets a complete CLE sample. In addition, we know that for a given $Y = |X|$, excursions of all $X^{(\beta,r)}$'s take place at the same times (i.e., at the same times as those of Y). Furthermore, we know that:

- For any given r, β, n and r_0 , the law of the loops corresponding to the first n excursions of $X^{(\beta,r)}$ that have length greater than r_0^2 , have the same law as part of a CLE_κ sample (in particular, they are disjoint simple loops almost surely).
- For any given β , the driving process $W^{(\beta,r)}$ converges uniformly to the driving function $W^{(\beta)}$ on any compact time interval, from which it follows that the loops corresponding to the excursions converge in Carathéodory topology.

We can therefore conclude that the excursions of $X^{(\beta)}$ correspond (via the Loewner transformation given by the driving function $W^{(\beta)}$) indeed to simple loops that are distributed like those of a CLE sample, and this completes the proof in the case where $\kappa < 4$.

For $\kappa = 4$ and $\mu \in \mathbb{R}$. In the same spirit, we choose the driving function

$$W_t^{<\mu,r>} := W_t + 4\mu r N_r(t) = 2B_t + 4(I_t^{(0)} + \mu r N_r(t)).$$

Then, the very same arguments as before show that the corresponding constructed loops are those of a CLE. And on the other hand, the driving function $W^{<\mu,r>}$ converges to $W^{<\mu>}$. This allows to complete the proof of the following fact:

Proposition 4. *When $\kappa = 4$, the law of $\text{CLE}_{4,\mu}(x)$ does depend neither on x nor on μ .*

4 The uniform exploration of CLE_4

Let us now modify the “symmetric” Loewner driving function $W^{(0)}$ by introducing some random jumps. Basically, at each time $T_n(r)$ (the end-times of the excursions of X of time-length at least r^2), we decide to resample the position of the driving function according to the uniform density in $[-m, m]$ – here m should be thought of as very large, we will then let it go to infinity.

Suppose for a while that m is fixed (we will omit to mention the dependence in m during the next paragraphs in order to avoid heavy notation). Let us associate to each excursion e of X a random variable $\xi(e)$ with this uniform distribution on $[-m, m]$, in such a way that conditionally on X , all these variables ξ are i.i.d. (for notational simplicity, we sometimes also write $\xi = \xi(T)$ when T is the time of X at which the corresponding excursion is finished). Then, we define the function $t \mapsto \hat{W}_t^r$ as follows: $T_0(r) = 0$ and for each $n \geq 0$,

- $\hat{W}^r(T_n) = \xi(T_{n+1})$.
- $\hat{W}^r - W^{(0)}$ is constant on each interval $[T_n(r), T_{n+1}(r))$.

The function \hat{W}^r is piecewise continuous, and it is therefore the driving function of some Loewner chain. The very same arguments as before show that for each given r , it defines a family of loops distributed like loops in a CLE_κ , and that completing the picture with independent CLE's in the not-yet-filled parts would give a full sample of CLE (note that the jump distribution – i.e. the choice of the new point according to the uniform distribution – is in fact independent of the future behavior of X).

It is easy to understand what happens to this construction when r tends to 0. As before, we are going to look at the almost sure behavior of \hat{W}^r when $r \rightarrow 0$, for a given sample of $W^{(0)}$ and ξ 's. Let us define the process \hat{W} by the fact that for each excursion e corresponding to a time-interval (S, T) ,

$$\hat{W}_t = (W_t^{(0)} - W_S^{(0)}) + \xi(T)$$

for $t \in [S, T)$ (this defines \hat{W} for all t , except on the zero-Lebesgue measure set of times that are not in the time-span of some excursion, for those times, we can choose \hat{W} as we wish). Then, clearly, the fact that $t \mapsto W_t^{(0)}$ is continuous ensures that for each given excursion interval, \hat{W}^r converges uniformly to \hat{W} as $r \rightarrow 0$ on this time-interval, because for small enough r , $\hat{W}^r = \hat{W}$ on this excursion. It follows readily that the Loewner chain generated by \hat{W}^r does (almost surely) converge (in Caratheodory topology) to the one generated by \hat{W} .

Hence, using the same arguments as above (the law of the traced loops is always that of loops in CLE, that are simple disjoint loops, the excursion-intervals correspond to the loops, and these intervals are the same for all r), we conclude that during each excursion time-interval, the driving process \hat{W} does indeed trace a loop, and that the joint law of all these loops are those of loops in a CLE.

Let us now rephrase all the above construction in terms of the Poisson point process of Bessel excursions $(e_u, u \geq 0)$. As we have explained earlier, each Bessel excursion in fact corresponds (via Loewner's equation) to a two-dimensional loop in the upper-half plane, that touches the boundary only at the origin. Let us call γ_u the loop corresponding to e_u . To each excursion e_u of the Bessel process, we also associate a random position $x_u \in \mathbb{R}$ sampled according to the uniform measure on $[-m, m]$ (more precisely, conditionally on all Bessel excursions (e_{u_j}) , the random variables (x_{u_j}) are i.i.d. with this distribution). Then, we define the loop $\hat{\gamma}_u$ by shifting γ_u horizontally by x_u (and so, the loop $\hat{\gamma}_u$ touches the real axis at x_u).

For each excursion e_u , we can now define the conformal transformation \hat{f}_u from the connected component of $\mathbb{H} \setminus \hat{\gamma}_u$ that contains i onto \mathbb{H} such that $\hat{f}_u(i) = i$ and $\hat{f}_u'(i) \in \mathbb{R}_+$. As this will be useful, we now reintroduce the dependence on m in the notation for these maps (and write $\hat{f}_u = \hat{f}_u^m$).

For a given m , we start with a Poisson point process $(e_u, u \geq 0)$ of Bessel excursions defined under the measure $2m\lambda$ and we then associate to each excursion the uniform random variable $\xi(e_u)$. A cleaner equivalent way to describe the process $((e_u, \xi_u), u \geq 0)$ is to say that it is a Poisson point process with intensity $\lambda \otimes dx 1_{x \in (-m, m)}$.

Then, clearly, we get a Poisson point process $(\hat{f}_u^m, u \geq 0)$ of such conformal maps (because for each u , \hat{f}_u^m is a deterministic function of the pair (e_u, x_u) and $((e_u, x_u), u \geq 0)$ is a Poisson point process).

For each $u > 0$, one can then define

$$\hat{F}_u^m = \circ_{v < u} \hat{f}_v^m$$

(where the composition is done in the order of appearance of the maps \hat{f}_v). Clearly, \hat{F}_u^m corresponds to the Loewner map (generated by the driving function \hat{W}) at the time (in the Loewner time parametrization) corresponding to the completion of all loops $\hat{\gamma}_v^m$ for $v < u$. In other words, if $\tau(e_u)$ is the time-length of the excursion e_u , the Loewner time at which the loop corresponding to that excursion will start being traced is $\sum_{v < u} \tau(e_v)$.

Hence, the loops

$$\tilde{\gamma}_u^m := (\hat{F}_u^m)^{-1}(\hat{\gamma}_u^m)$$

are distributed like CLE loops. In particular, the loop that contains i will be the loop $(\hat{F}_\tau^m)^{-1}(\hat{\gamma}_\tau^m)$ where

$$\tau = \inf\{u \geq 0 : \hat{\gamma}_u^m \text{ surrounds } i\}.$$

Let us rephrase what we have done so far: For each m , we have seen that one can define CLE loops by considering the Loewner chain generated by \hat{W} , using the Poisson point process $(\hat{f}_u^m, u \geq 0)$, or equivalently, via the Poisson point process $\hat{\Gamma}^m := (\hat{\gamma}_u^m, u \geq 0)$ with intensity measure

$$M^m = \int_{-m}^m dx \mu^x$$

where μ^x denotes the measure on loops rooted at x (like in [11], we define this measure as the measure μ^0 on loops in the upper half-plane generated via a Bessel excursion defined under λ , and shifted horizontally by x).

Now, let us describe what happens when $m \rightarrow \infty$. Suppose now that we consider the Poisson point process $\hat{\Gamma} := (\hat{\gamma}_u, u \geq 0)$ with intensity

$$M := \int_{\mathbb{R}} dx \mu^x$$

and the corresponding iterations of maps \hat{F}_u . Even though this measure seems “even more infinite” than M^m , this iteration of conformal maps does not explode. This is due to the scaling properties of μ^x and to the fact that one normalizes always at i (so that loops rooted far away do not contribute much the derivative at i) – one can for instance justify this using Lemma 6 below.

Note also that if we keep only those loops in $\hat{\Gamma}$ that are rooted at a point in $[-m, m]$, we obtain a process with the same law as $\hat{\Gamma}^m$. The key observation is now to see that when $m \rightarrow \infty$, each map \hat{F}_u^m converges uniformly in any compact subdomain of the closed upper half-plane to \hat{F}_u . This implies the following:

Lemma 5. *The loops $\tilde{\Gamma} = (\tilde{\gamma}_u := \hat{F}_u^{-1}(\hat{\gamma}_u), u \leq \tau)$ are also distributed like loops in a CLE.*

At this stage, everything we have said is still true if we replace the Lebesgue measure on \mathbb{R} by (almost) any other given distribution on \mathbb{R} , and any $\kappa \in (8/3, 4]$. An important reason to choose this particular measure and to focus on the case where $\kappa = 4$ is that the following Lemma holds only in this case:

Lemma 6. *When $\kappa = 4$ the measure M is conformal invariant.*

Proof. Recall from [11] that when $\kappa = 4$ and if Φ is a conformal transformation of the half-plane onto itself,

$$\Phi \circ \mu^x = |\Phi'(x)| \mu^{\Phi(x)}$$

where the measure $\Phi \circ \mu^x$ is defined by

$$\Phi \circ \mu^x(A) = \mu^x\{\gamma : \Phi(\gamma) \in A\}.$$

Hence, it follows immediately that $\Phi \circ M = M$. \square

A direct consequence of this conformal invariance is that

Corollary 7. *When $\kappa = 4$, the law of $\tilde{\Gamma} = (\tilde{\gamma}_u, u \leq \tau)$ is invariant under any Moebius transformation Φ of the upper half-plane that preserves i .*

Note that there is no time-change involved. The law of $\Phi(\tilde{\gamma}_u)1_{u \leq \tau}$ and $\tilde{\gamma}_u 1_{u \leq \tau}$ are for instance identical.

Proof. Let Φ be a Moebius transformation of the upper half-plane that preserves i , and $(\hat{\gamma}_u, u \geq 0)$ be a Poisson point process with intensity M . And define τ, \hat{f}_u for $u < \tau$, and $\hat{F}_u, \tilde{\gamma}_u$ for $u \leq \tau$ as described above.

Note that $(\bar{\gamma}_u := \Phi(\hat{\gamma}_u), u \geq 0)$ is a Poisson point process with intensity $M = \Phi \circ M$, and it has therefore the same distribution as $(\hat{\gamma}_u, u \geq 0)$. For $u < \tau$, let \bar{f}_u be the conformal map from the connected component of $\mathbb{H} \setminus \bar{\gamma}_u$ that contains i onto \mathbb{H} such that $\bar{f}_u(i) = i$ and $\bar{f}'_u(i) \in \mathbb{R}_+$. It is easy to see that

$$\bar{f}_u = \Phi \circ \hat{f}_u \circ \Phi^{-1}$$

and hence for $u \leq \tau$,

$$\bar{F}_u := \circ_{v < u} \bar{f}_v = \Phi \circ \hat{F}_u \circ \Phi^{-1}.$$

As a result, for $u \leq \tau$,

$$\Phi(\tilde{\gamma}_u) = \Phi(\hat{F}_u(\hat{\gamma}_u)) = \bar{F}_u(\bar{\gamma}_u).$$

Since $(\bar{\gamma}_u, u \geq 0)$ has the same distribution as $(\hat{\gamma}_u, u \geq 0)$, it follows that $(\Phi(\tilde{\gamma}_u), u \leq \tau)$ has the same distribution as $(\tilde{\gamma}_u, u \leq \tau)$. \square

In fact, a stronger result holds. Let us now choose some other point z than i in the upper half-plane. Let σ denote the first moment if it exists at which the process $(\tilde{\gamma}_u, u \leq \tau)$ disconnects i from z . If the loop $\tilde{\gamma}_\tau$ surrounds both i and z , we simply set $\sigma = \tau$. Note that the event that $\sigma < \tau$ can happen when the process discovers a loop surrounding one of the two points and not the other, but at this stage, it is not excluded that it can disconnect two points strictly before discovering the loops that surround them, just like the symmetric SLE($\kappa, \kappa - 6$) does, see [11].

Define the same process $(\tilde{\gamma}_u^z, u \leq \tau^z)$ as above, except that we choose to normalize “at z ” instead of normalizing at i . One way to describe it would be to write $\tilde{\gamma}_u^z = \varphi(\tilde{\gamma}_u)$, where φ is the affine transformation from \mathbb{H} onto itself such that $\varphi(i) = z$ (but we will use another way to describe it in terms of $\tilde{\gamma}_u$ in a moment). We then define σ^z to be the first moment at which it disconnects z from i or discovers the loop that surrounds both points.

Lemma 8. *When $\kappa = 4$, and for any $z \in \mathbb{H}$, the law of $(\tilde{\gamma}_u^z, u \leq \sigma^z)$ is identical to the law of $(\tilde{\gamma}_u, u \leq \sigma)$.*

We shall use the following classical result about Poisson point process (see for instance [1], Section 0.5):

Result. *Let $(a_u, u \geq 0)$ be a Poisson point process with some intensity ν (defined in some metric space A). Let $\mathfrak{F}_{u-} = \sigma(a_v, v < u)$. If $(\Phi_u, u \geq 0)$ is a process (with values on functions of A onto A) such that for any $u \geq 0$, Φ_u is \mathfrak{F}_{u-} -measurable, and that Φ_u preserves ν then $(\Phi_u(a_u), u \geq 0)$ is still a Poisson point process with intensity ν .*

We are now ready to prove the lemma.

Proof. Consider the process $(\tilde{\gamma}_u, u \leq \tau)$ defined from the Poisson point process $(\hat{\gamma}_u, u \geq 0)$ with intensity M as above (and keep the same definitions for τ , \hat{f}_u with $u < \tau$, $\hat{F}_u, \tilde{\gamma}_u$ with $u \leq \tau$, where the latter are defined by normalizing the maps at i).

We denote $\mathfrak{F}_{u-} = \sigma(\hat{\gamma}_v, v < u)$. For $u < \sigma$, $\hat{F}_u^{-1}(\mathbb{H})$ is a simply connected domain of \mathbb{H} containing z . Let G_u be the conformal map from $\hat{F}_u^{-1}(\mathbb{H})$ onto \mathbb{H} normalized at z by $G_u(z) = z$ and $G'_u(z) \in \mathbb{R}_+$. Define for each $u < \sigma$

$$\Phi_u = G_u \circ \hat{F}_u^{-1}.$$

We also define $\Phi_\sigma = \lim_{u \rightarrow \sigma-} \Phi_u$, and we say that Φ_u is the identity map for all $u > \sigma$. It is clear that, for each positive u , the map Φ_u is a \mathfrak{F}_{u-} -measurable Moebius transformation from the upper half-plane onto itself. Hence, the process $(\bar{\gamma}_u := \Phi_u(\hat{\gamma}_u), u \geq 0)$ is also Poisson point process with intensity M .

If we use the point process $(\bar{\gamma}_u)$ to construct the process $(\tilde{\gamma}_u^z)$ normalized at z , we get a coupling of $(\tilde{\gamma}_u)$ and $(\tilde{\gamma}_u^z)$ in such a way that they coincide up to time σ : For all $u < \sigma$, $\tilde{\gamma}_u = \tilde{\gamma}_u^z$, and in addition,

$$\bar{\gamma}_\sigma = \Phi_\sigma(\hat{\gamma}_\sigma)$$

(if this $\hat{\gamma}_\sigma$ exists) so that $\tilde{\gamma}_\sigma = \tilde{\gamma}_\sigma^z$.

Hence, with this coupling, we see that $\sigma \leq \sigma^z$ almost surely. By symmetry (because there exists a conformal map interchanging these two points), it follows that $\sigma = \sigma^z$ almost surely. \square

This means that it is possible to couple these two processes up to the first moment at which it disconnects i from z . By scaling, this shows that for any pair of points z and z' , we can couple the two processes $\tilde{\gamma}^z$ and $\tilde{\gamma}^{z'}$ up to the first time at which they disconnect z from z' . Hence, it is possible to couple the processes $\tilde{\gamma}^z$ for *all* $z \in \mathbb{H}$ simultaneously in such a way that for any two points z and z' , the previous statement holds.

If we now use such a coupling, we get a Markov process on domains $(D_u, u \geq 0)$: At time $u = 0$, the domain is the upper half-plane, and at time $u > 0$, it is the union of all the (disjoint) open sets corresponding to the evolution to all points z at time u . Existence of such a conformally invariant process is a rather striking feature, as it uses no reference point, and the time of the evolution is preserved through the conformal transformation. The processes $(D_u, u \geq 0)$ and $(\Phi(D_u), u \geq 0)$ are identically distributed (with no time-change) for all Moebius transformations Φ .

Note that this uniform exploration mechanism can also be viewed as the limit (in law) of the asymmetric $\text{CLE}_{4,\mu}$ construction of Proposition 4 in the limit when $\mu \rightarrow \infty$ (and the boundary points $+\infty$ and $-\infty$ of the upper half-plane are identified, alternatively, one can state this easily in the radial setting). We leave the details to the interested reader.

5 Comments and open questions

5.1 Some open questions.

We first mention some natural open questions that are closely related to the present paper:

It is proved in [6, 3] that an SLE_4 can be deterministically drawn as the contour lines (or sometimes called level-lines or cliff-lines) in a Gaussian Free Field with appropriate boundary conditions. See also [5] for the fact that the entire CLE_4 can be deterministically embedded in a Gaussian Free Field. Note that the symmetric exploration process is more naturally associated to the Gaussian Free Field than the asymmetric ones, because when one defines a Gaussian Free Field out of a CLE_4 , one has to toss an independent coin for each CLE_4 loop to decide an orientation, so that the symmetric $\text{SLE}(4, -2)$ (including the coin tosses) is defined via the randomness present in the GFF.

In the present paper, we described various conformally invariant ways to construct a CLE, via Loewner chains. This induces additional information than just the CLE (which loop is discovered where, what is the starting and end-point of the loop when one uses a given exploration etc.).

1. If we are given a CLE_κ in a simply connected domain D and a starting point on the boundary of D , and a family of Bernoulli random variables $\epsilon(\gamma)$ with parameter β (one for each CLE loop γ), is the asymmetric exploration process with parameter β deterministically defined?
2. In particular, is the totally asymmetric exploration process (when $\beta = 1$) sample a deterministic function of the CLE sample and of the starting point?
3. Is the uniform exploration process in fact a deterministic function of the CLE_4 ?

A positive answer to this last question would give rise to a conformally invariant distance between loops in a CLE_4 . We plan to investigate further this “distance” between loops in a CLE_4 in a forthcoming paper [12].

5.2 Discrete explorations

Our proofs rely a lot on the fact that the symmetric Bessel explorations do indeed construct the loops in a CLE, which was derived in [11] using a discretization of the exploration procedure that was proved to converge to the symmetric Bessel construction. The other CLE constructions that we have studied in the present paper also have natural discrete counterparts that we now briefly describe. However, it turns out to be (seemingly) technically more unpleasant to control the convergence of these asymmetric discrete exploration procedures

than the symmetric ones, so that it seemed simpler to derive our results building on the relation between CLE's and the symmetric construction.

We first recall the exploration procedures described in [11] to explore a CLE little by little (here a CLE is just a collection of loops that satisfy the CLE axioms defined in [11]). It will be easier to explain things in the radial setting i.e. in the unit disk instead of the half-plane.

Suppose that $\Gamma = (\gamma_j, j \in J)$ is a CLE in the unit disc \mathbb{U} and that $\epsilon > 0$ is given. We denote $\gamma(z)$ as the loop of Γ (when it exists) surrounding $z \in \mathbb{U}$. Throughout this section, $D(1, \epsilon)$ will denote the image of the set $\{z \in \mathbb{H} : |z| < \epsilon\}$ under the conformal map $\Psi : z \mapsto (i - z)/(i + z)$ from the upper half-plane \mathbb{H} onto the unit disc such that $\Psi(i) = 0, \Psi(0) = 1$. Note that for small ϵ , this set is rather close to a small semi-disc centered at 1.

At the first step, we “explore” the small shape $D(1, \epsilon)$ in \mathbb{U} , and we discover all the loops in Γ that intersect $D(1, \epsilon)$. If $\gamma(0)$ has already been discovered during this first step, we define $N = 1$ and we stop. Otherwise, we let U_1 denote the connected component that contains the origin of the set obtained when removing from $U'_1 = \mathbb{U} \setminus D(1, \epsilon)$ all the loops that do not stay in U'_1 . From the restriction property in the CLE axioms, the conditional law of Γ restricted to U_1 (given U_1) is that of a CLE in this domain.

We now choose some point x_1 on ∂U_1 , and the conformal map φ_1^ϵ from U_1 onto \mathbb{U} such that $\varphi_1^\epsilon(0) = 0$ and $\varphi_1^\epsilon(x_1) = 1$. Note that we allow here for different possible choices for x_1 . It can be a deterministic function of U_1 , but the choice of x_1 can also involve additional randomness (we can for instance choose it according to the harmonic measure at the origin etc.), but we impose the constraint that conditionally on U_1 , the CLE restricted to U_1 and the point x_1 are conditionally independent (in other words, one is not allowed to use information about the loops in U_1 in order to choose x_1).

During the second step of the exploration, one discovers the loops of $\Gamma_1 := \varphi_1^\epsilon(\Gamma \cap U_1)$ that intersect $D(1, \epsilon)$. In other words, we consider the pushforward of Γ by φ_1^ϵ (which has the same law as Γ itself, due to the CLE axioms) and we repeat step 1. If we discover a loop that surrounds the origin at that step, then we stop and define $N = 2$. Otherwise, we define the connected component U_2 that contains the origin of the domain obtained when removing from $\mathbb{U} \setminus D(1, \epsilon)$ the loops of Γ_1 that do not stay in this domain, and we define the conformal map φ_2^ϵ from U_2 onto \mathbb{U} with $\varphi_2^\epsilon(0) = 0$ and $\varphi_2^\epsilon(x_2) = 1$, where x_2 is chosen in a conditionally independent way of $\Gamma_1 \cap U_2$, given U_1, U_2 and x_1 .

We then explore $\Gamma_2 := \varphi_2^\epsilon(\Gamma_1)$ and so on. We can iterate this procedure until the step N at which we eventually “discover” a loop that surrounds the origin. Note that $\gamma(0)$ (the loop in Γ that surrounds the origin) is the preimage of this loop (the loop in Γ_N that surrounds the origin and intersects $D(1, \epsilon)$) under $\varphi_N^\epsilon \circ \dots \circ \varphi_1^\epsilon$.

In this definition, the discrete exploration “strategy” is encoded by ϵ (the “step-size”) and by the rule used to choose the x_n 's. Since the probability to discover the loop at each given step n (conditionally on the fact that it has not been discovered before) is constant and positive, it follows that N is almost surely finite, that its law is geometric (regardless of the choice of x_n 's).

In [11], it is shown that if a CLE (ie. satisfying the CLE axioms exist), then its loops are of SLE_κ -type for some $\kappa \in (8/3, 4]$ and that it is necessarily the one constructed via

the symmetric Bessel construction (and therefore unique). Conversely (using a different argument involving loop-soups) it is shown that these CLE do exist. The strategy of one part of the proof is to control the behavior of certain natural discrete exploration strategies when ϵ tends to 0:

The first one is the “exploration normalized at the origin”. Here, at each step, x_n and φ_n^ϵ are chosen according to the rule that $(\varphi_n^\epsilon)'(0)$ is a positive real number. In other words, we choose φ_n^ϵ using the standard normalization at the origin.

Towards the end of the paper [11], it is shown that the symmetric exploration indeed constructs an axiomatic CLE, by using the following symmetric discrete exploration procedure: Define 1_ϵ^+ and 1_ϵ^- the two intersections of $\partial D(1, \epsilon)$ with the unit circle. At each step, one tosses a (new) fair coin to decide which one of the two points gets mapped conformally onto 1. In other words, the maps φ_n^ϵ are i.i.d., x_1 is independent of U_1 , and

$$P(x_1 = 1_\epsilon^+) = P(x_1 = 1_\epsilon^-) = 1/2.$$

The definition of the asymmetric discrete explorations is then natural: For a given β , we toss a $(1 + \beta)/2$ vs. $(1 - \beta)/2$ coin in order to choose which one of the two points 1_ϵ^+ or 1_ϵ^- to choose, but in order to compensate the created bias, we post-compose the obtained map $\tilde{\varphi}_n^\epsilon$ with a deterministic rotation of some angle $\theta(\epsilon)$ that vanishes as $\epsilon \rightarrow 0$ (that corresponds to the jump in the approximation $I^{(\beta, r)}$ of $I^{(\beta)}$). However, we see that this rotation depends on the chosen base-point (here the origin); this is one reason for which this discrete approximation is a little harder to master than in the case $\beta = 0$.

The definition of uniform discrete approximations is also very natural: Just choose x_n at random on the boundary of U_n according to the harmonic measure seen from 0. Equivalently, choose any φ_n^ϵ and compose it with a uniformly chosen rotation. Again, this rule depends on the target point (the origin) – but this one is less tricky to control as $\epsilon \rightarrow 0$. We leave it to the interested (and motivated) reader to check that these discrete explorations indeed converge in distribution to the continuous CLE constructions that we have studied in the present paper.

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